Bayesian control chart for nonconformities

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Abstract

The c - chart or the control chart for nonconformities is designed for the case where one deals with the number of defects or nonconformities observed. A control chart can be developed for the total or average number of nonconformities per unit, which is well modelled by the Poisson distribution. In this paper the c - chart will be studied, where the usual operation of the c - chart will be extended by introducing a Bayesian approach for the c - chart. Control chart limits, average run lengths and false alarm rates will be determined by using a Bayesian method. These results will be compared to the results obtained when using the classical (frequentist) method.

Key words: Bayesian analysis; c - chart; Poisson distribution; Posterior predictive density; Tolerance interval

1 Introduction

In this paper the control chart for nonconformities, also known as the c - chart, will be studied. The c - chart or the control chart for nonconformities is designed for the case where one deals with the number of defects or nonconformities observed. A control chart can be developed for the total or average number of nonconformities per unit, which is well modelled by the Poisson distribution. In most of the standard textbooks on Quality Control, see for example Montgomery (1996), the Poisson parameter is indicated by c, in this paper the Poisson parameter will be indicated by λ.

If there is no standard given value, then λ should be estimated as the average number of nonconformities in an initial sample. This average number of nonconformities will be indicated by $\bar{\lambda}$. From Montgomery (1996) the control chart will be defined as:

$$\begin{aligned} \text{UCL} &= \bar{\lambda} + 3\sqrt{\bar{\lambda}} \\ \text{Centre line} &= \bar{\lambda} \\ \text{LCL} &= \bar{\lambda} - 3\sqrt{\bar{\lambda}}. \end{aligned}$$

The above mentioned control chart is when the value of λ is unknown, and is the well known classical (frequentist) method. This case is also referred to as the “no standard given” case. When the parameter is unknown, the common practice is to estimate the parameter from phase I of the study. Once the control chart is set, independent inspection units are selected, and the number of nonconformities in the each inspection unit is determined and plotted on the chart. If a point falls within the lower and upper control limit, the process is in-control. If a point falls outside or on the lower or upper control limit, the process is out-of-control. When this happens, an alarm or a signal is given. The standard (frequentist) methods for statistical process control (SPC) depend on long-run stability to establish a pattern against which further samples may be compared, (Woodward & Naylor, 1993).

Menzefricke (2002) proposed a Bayesian approach to obtain control charts, where a predictive distribution based on a Bayesian approach is used to derive the rejection region and to the construct the control chart. Menzefricke (2002) applied this to the control chart for means with known standard deviation and to means with unknown standard deviation. Calabrese (1995) considered a process control procedure with fixed sample sizes and sampling intervals, where a Bayesian model is developed for process control under standard cost and operating assumptions. Calabrese (1995) states that Taylor (1965) and Taylor (1967) showed that non-Bayesian techniques are not optimal and suggests that action decisions, sampling size and frequency should be determined based on the posterior probability that the process has shifted to an out-of-control state. Bayarri & García-Donato (2005) focused on u-charts, where one is interested in the number of nonconformities per inspection unit. When the inspection unit is constant over time, this chart is referred to as the c-chart. Bayarri & García-Donato (2005) extended the usual operation of the u - chart, by introducing an empirical Bayesian model and a Bayesian sequential approach. They highlight the following concerns with using the classical approach:

- The Poisson model is often a very poor fit to this type of data;
- there is quite a long period in which the process is not controlled at all, namely the one used to estimate the parameter (i.e. phase I);
- previous information cannot be incorporated in any way.

We will introduce a Bayesian approach for the c - chart. The predictive density will be used to obtain the control chart. Control chart limits, average run lengths and false alarm rates will be determined by using a Bayesian method. These results will be compared to the results obtained when using the classical (frequentist) method. We will consider an objective Bayesian approach, i.e. we will make use of noninformative priors. Tolerance intervals will also be introduced, since the application of tolerance intervals can be useful in quality control. A tolerance interval gives information about a certain proportion or more of the population, with a given confidence level. This proportion is also referred to as the content of a tolerance interval. Whereas a confidence interval gives information about an unknown parameter. The goal of a tolerance interval is to contain at least a specified proportion of the population with a specified degree of confidence. For control charts, one can argue that the probability content outside of the control limits needs to be controlled, for example 0.00135 below the lower control limit and 0.00135 above the upper control limit, (Hamada, 2002).

In Section 2 the choice of prior, the posterior distribution and the predictive density will be discussed, a simulation study will be discussed in Section 3 and an example will be considered in Section 4. Tolerance intervals will briefly be discussed in Section 5 and some concluding remarks will discussed in Section 6.

2 Prior Distribution, Posterior Distribution and Predictive density

For this study we will consider an objective Bayesian approach. Yang & Berger (1997) listed the following noninformative priors for the Poisson distribution: uniform, Jeffreys and reference priors. The probability matching prior is another noninformative prior which can be considered. From Yang & Berger (1997) the Jeffreys prior is the same as the reference prior for this case. The Jeffreys prior, is proportional to the square root of the determinant of the Fisher information matrix and is given by

$$\pi_R(\lambda) = \pi_J(\lambda) \quad \propto \quad \lambda^{-\frac{1}{2}}. \tag{1}$$

The probability matching prior for the product of different powers of k Poisson rates, $\prod_{i=1}^{k} \lambda_i^{\alpha_i}$, is given by $\pi_{PM}(\underline{\lambda}) \propto \left(\sum_{i=1}^{k} \lambda_i^{-1} \alpha_i^2\right)^{\frac{1}{2}}$. See Kim (2006) and Raubenheimer & Van der Merwe (2012) for further discussion. If we let $k = 1$ and $\alpha = 1$, the probability matching prior is given by

$$\pi_{PM}(\lambda) \quad \propto \quad \lambda^{-\frac{1}{2}}. \tag{2}$$

From Equations 1 and 2 it is clear that the Jeffreys, reference and probability matching priors yield the same prior.

The uniform prior is proportional to a constant and is given by

$$\pi_U(\lambda) \quad \propto \quad \text{constant}. \tag{3}$$

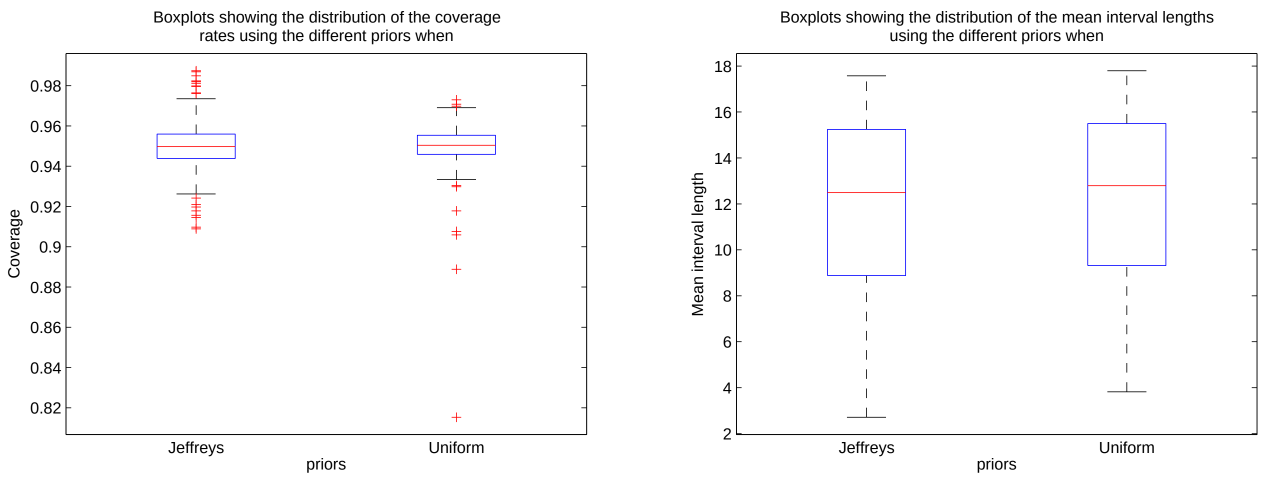

Figure 1: Coverage rates and average interval lengths.

In Table 1, average coverage probabilities, mean lengths and standard deviations are given for $\lambda = 0.1:0.1:20$. The averages are taken over the different values of λ. These results are also summarised in Figure 1, where boxplots are constructed for the coverage rates and the average interval lengths. MATLAB® was used to construct the box plots in this chapter. On each box, the central mark is the median, the edges of the box are the 25^{th} and 75^{th} percentiles, the whiskers extend to the most extreme data points the algorithm considers not to be outliers, and the outliers are plotted individually.

Table 1: Averages over the different values of λ for 95% credibility intervals for λ.

	$\lambda = 0.1:0.1:20$	
	Jeffreys prior	**Uniform prior**
Coverage	0.9502	0.9491
Average interval lengths	11.8281	12.2147
Standard deviation	1.9176	1.8368

The performances of the priors are very similar, and therefore we decided to use the Jeffreys prior which is also the probability matching prior in this case. Ghosh (2011) states that the uniform prior has often been criticised due to its lack of invariance under one-to-one reparameterisation and that the Jeffreys prior is invariant under one-to-one reparameterisation of parameters.

The predictive density will be used to obtain the control chart. If independent inspection units are randomly selected at equally spaced time intervals, the number of nonconformities in the i^{th} inspection will follow a Poisson distribution with parameter λ. Therefore $P(X_i = x_i) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$ for $x_i = 0, 1, 2, \ldots$ and $i = 1, \ldots, m$. The likelihood function will thus be

$$L(\lambda \,|data) \;\propto\; e^{-m\lambda} \lambda^{\sum_{i=1}^{m} x_i}. \qquad (4)$$

As mentioned the Jeffreys prior will be used, which is given by

$$\pi_J(\lambda) \;\propto\; \lambda^{-\frac{1}{2}}. \tag{5}$$

Combining Equations 4 and 5 it follows that the posterior distribution of λ is a $Gamma\left(\sum_{i=1}^{m} x_i + \frac{1}{2}, m\right)$ distribution, i.e.

$$\pi_J(\lambda\,|data) \;=\; \frac{m^{\sum_{i=1}^{m} x_i+\frac{1}{2}}}{\Gamma\left(\sum_{i=1}^{m} x_i + \frac{1}{2}\right)} e^{-m\lambda} \lambda^{\sum_{i=1}^{m} x_i - \frac{1}{2}} \qquad \lambda > 0. \tag{6}$$

Denote the number of nonconformities in a future inspection unit by x_f, then the predictive density (if λ is known) is

$$f\left(x_f\,|\lambda\right) \;=\; e^{-\lambda}\frac{\lambda^{x_f}}{x_f!} \qquad x_f = 0, 1, \ldots, \infty \tag{7}$$

and the unconditional predictive density is

$$\begin{aligned} f\left(x_f\,|data\right) &= \int_0^{\infty} f\left(x_f\,|\lambda\right)\pi_J(\lambda\,|data)\,d\lambda \\ &= \frac{m^{\sum_{i=1}^{m} x_i+\frac{1}{2}}}{\Gamma\left(\sum_{i=1}^{m} x_i + \frac{1}{2}\right) x_f!} \frac{\Gamma\left(x_f + \sum_{i=1}^{m} x_i + \frac{1}{2}\right)}{(m+1)^{x_f + \sum_{i=1}^{m} x_i + \frac{1}{2}}} \qquad x_f = 0, 1, \ldots, \infty \end{aligned} \tag{8}$$

which is a Poisson-gamma distribution whose mean and standard deviation are easily derived. It is known as a Poisson-gamma distribution, because it is generated by the mixture of Poisson and gamma distributions.

Equation 8 can be written as

$$f\left(x_f\,|data\right) \;=\; \frac{\Gamma\left(x_f + \sum_{i=1}^{m} x_i + \frac{1}{2}\right)}{\Gamma\left(\sum_{i=1}^{m} x_i + \frac{1}{2}\right)\Gamma\left(x_f+1\right)} \left(\frac{m}{m+1}\right)^{\sum_{i=1}^{m} x_i+\frac{1}{2}} \left(\frac{1}{m+1}\right)^{x_f} \qquad x_f = 0, 1, \ldots, \infty \tag{9}$$

which is a negative binomial distribution with mean $\frac{\sum_{i=1}^{m} x_i+\frac{1}{2}}{m}$ and variance $\frac{\sum_{i=1}^{m} x_i+\frac{1}{2}}{m^2}(m+1)$. The posterior predictive distribution has the same mean as the posterior distribution, but a greater variance. We re-wrote the Poisson-gamma distribution in Equation 8 as a negative binomial distribution given in Equation 9, since it is easier to simulate from a negative binomial distribution than a Poisson-gamma distribution in MATLAB®. MATLAB® has a built-in function for the negative binomial distribution in the statistical toolbox, but not for the Poisson-gamma distribution. The predictive distribution in can be used to obtain the control chart limits. The size of the rejection region, $R^{\star}(\alpha)$, is then defined as

$$\alpha = \sum_{R^{\star}(\alpha)} f\left(x_f|data\right).$$

If a point falls within the lower and upper control limit, the process is in-control. If a point falls outside or on the lower or upper control limit, the process is out-of-control. When this happens, an alarm or a signal is given. The probability of a "signal" when the process is in-control, is also known as the false alarm rate (FAR).

Assuming that the process remains stable, the predictive distribution can be used to derive the distribution of the run lengths. Given λ and a stable process, the distribution of the run length $r^{\star}$ is geometric with parameter FAR, the probability of a "signal". The average run length (ARL) is calculated as:

$$\text{ARL} = \frac{1}{P\left(\text{sample point plots out of control}\right)}.$$

If the process is in-control, the expected nominal value for the false alarm rate is 0.0027 and the expected nominal value for the average run length is $(0.0027)^{-1} = 370.3704$.

3 Simulation Study

Our aim in this simulation study is to compare the unconditional average run lengths and unconditional false alarm rates using the frequentist method and the proposed Bayesian method. Lower and upper control limits will be calculated for given m and λ values. The values used for m and λ are the same as the values used by Chakraborti & Human (2008).

The predictive density given in Equation 8 will be used to obtain the control chart limits when using the Bayesian approach. The predictive density is a Poisson-gamma distribution with parameters $\sum_{i=1}^{m} x_i + \frac{1}{2}$ and m.

Consider the following simulation study where λ =1, 2, 4, 6, 8, 10, 20, 50 and m =5, 10, 15, 20, 25, 30, 50, 100, 200, 300, 500, 1 000. The number of simulations is equal to 10 000. Chakraborti & Human (2008) considered these values and used the classical (frequentist) method to obtain the unconditional false alarm rates and the unconditional average run lengths. We will consider these values and apply the proposed Bayesian method to obtain the unconditional false alarm rates and the unconditional average run lengths. The results for the given m and λ values, using the classical and the Bayesian methods, are given in Tables 2 and 3.

A larger value for the run length is desired. As stated by Chakraborti & Human (2008), the performance of a control chart is usually judged on the basis of its run length distribution. The run length distribution is the probability distribution of the random variable which denotes the number of inspection units that must be sampled before the first signal is observed on the chart. From Table 2 it can be seen that the Bayesian method yields larger values for the average run length for most of the choices of m and λ, except for $\lambda-$values of 8, 10, and 20. A smaller value for the false alarm rate is desired, since the false alarm rate is the probability of a signal being given by the control chart when the process is actually in-control. From Table 3 it can be seen that the Bayesian method yields smaller values for the false alarm rates for most of the choices of m and λ, except for $\lambda-$values of 2 (when m is small) and 10 (when m is large).

Table 2: Unconditional average run lengths using the frequentist and Bayesian methods for different values of m and λ.

λ	Freq	Bayes	Freq	Bayes	Freq	Bayes	Freq	Bayes
	$m=5$		$m=10$		$m=15$		$m=20$	
1	2.5236	2.9726	2.592	2.844	2.6185	2.8019	2.6286	2.7803
2	6.603	7.9912	6.8337	7.6953	6.8798	7.5813	6.9335	7.5299
4	39.3789	53.8293	40.9348	53.4764	41.4612	52.9537	41.8947	52.9792
6	166.7599	185.8763	162.0944	217.3853	159.7376	236.0023	158.2355	250.0925
8	455.0527	230.1522	384.2628	232.7177	334.809	232.5899	318.8482	236.3737
10	400.8703	257.197	381.928	265.7485	355.9958	262.1421	354.2223	264.1818
20	302.3074	323.7619	331.2267	318.6245	333.4108	321.8224	337.9252	318.2154
50	257.9259	346.8771	295.8393	348.6708	313.1295	349.4466	321.1752	349.4278
	$m=25$		$m=30$		$m=50$		$m=100$	
1	2.6338	2.767	2.6364	2.7581	2.6391	2.7394	2.6393	2.7261
2	6.9353	7.4877	6.9625	7.4639	6.9875	7.4226	7.0322	7.3899
4	42.1259	52.9556	42.155	52.9015	42.4984	52.6893	42.558	52.3333
6	156.854	260.7488	155.8762	270.4585	153.8603	284.1739	154.1287	297.1448
8	300.4126	237.0183	291.0695	241.8632	275.3686	249.3841	261.4704	254.6228
10	345.129	259.5524	332.8875	259.3479	322.339	264.2547	307.7115	268.811
20	336.748	318.0466	335.0982	319.3534	334.5603	320.2731	334.1226	317.6258
50	327.0982	349.8278	331.5268	350.5079	338.7516	349.5367	345.7904	349.9675
	$m=200$		$m=300$		$m=500$		$m=1000$	
1	2.6395	2.7212	2.639	2.7178	2.6387	2.7158	2.6382	2.7144
2	7.0807	7.3588	7.1083	7.35	7.1317	7.3422	7.147	7.3362
4	42.5555	52.1365	42.5557	52.0145	42.5568	51.9554	42.5521	51.9368
6	156.7386	304.2012	159.1996	305.9919	161.8554	307.2618	163.5382	309.4095
8	252.2631	257.9758	248.8586	260.7379	247.0592	263.8656	246.6917	267.8421
10	295.1461	272.3508	290.1577	275.0636	286.7668	278.7787	285.7598	283.2577
20	333.2341	321.1212	333.4213	323.8763	334.7567	328.2414	338.0604	333.5165
50	349.4351	351.4026	351.1822	354.3027	355.8243	359.8573	366.8489	367.4067

From this simulation study we can conclude that the Bayesian method generally gives larger average run lengths than the classical method. This is the case for most of the choices of m and λ, except for $\lambda-$values of 8, 10, and 20 when considering the average run length. After careful investigation of Table

2, it is noticeable for the classical method, that there is a big increase in the average run lenghts when λ is equal to 8 and 10, and then the average run length decreases when λ is equal to 20 and 50. For the Bayesian method, the average run length increases consistently as the value of λ increases.

Table 3: Unconditional false alarm rates using the frequentist and Bayesian methods for different values of m and λ.

λ	Freq	Bayes	Freq	Bayes	Freq	Bayes	Freq	Bayes
	$m=5$		$m=10$		$m=15$		$m=20$	
1	0.4041	0.3973	0.3881	0.3848	0.3828	0.3796	0.3810	0.3770
2	0.1570	0.1734	0.1479	0.1555	0.1462	0.1492	0.1448	0.1459
4	0.0318	0.0333	0.0269	0.026	0.0257	0.0237	0.0249	0.0224
6	0.0131	0.0079	0.0096	0.0058	0.0086	0.0052	0.0082	0.0048
8	0.0104	0.0052	0.0068	0.0051	0.0059	0.0050	0.0054	0.0049
10	0.0092	0.0043	0.0060	0.0042	0.0053	0.0043	0.0047	0.0042
20	0.0078	0.0032	0.0051	0.0032	0.0045	0.0032	0.0041	0.0032
50	0.0068	0.0029	0.0046	0.0029	0.004	0.0029	0.0037	0.0029
	$m=25$		$m=30$		$m=50$		$m=100$	
1	0.3715	0.3755	0.3796	0.3745	0.3791	0.3722	0.3790	0.3704
2	0.1399	0.1441	0.1440	0.1428	0.1433	0.1400	0.1424	0.1380
4	0.0212	0.0218	0.0245	0.0213	0.0240	0.0204	0.0238	0.0198
6	0.0061	0.0045	0.0077	0.0042	0.0074	0.0039	0.0070	0.0035
8	0.0041	0.0048	0.0050	0.0046	0.0047	0.0043	0.0044	0.0041
10	0.0035	0.0042	0.0044	0.0042	0.0040	0.0041	0.0038	0.0039
20	0.0029	0.0033	0.0038	0.0033	0.0035	0.0032	0.0033	0.0032
50	0.0029	0.0029	0.0034	0.0029	0.0032	0.0029	0.0030	0.0029
	$m=200$		$m=300$		$m=500$		$m=1000$	
1	0.3790	0.3693	0.3791	0.3691	0.3791	0.3689	0.3792	0.3688
2	0.1413	0.1373	0.1407	0.1370	0.1402	0.1367	0.1399	0.1365
4	0.0238	0.0195	0.0238	0.0194	0.0238	0.0194	0.0238	0.0193
6	0.0066	0.0034	0.0064	0.0034	0.0062	0.0033	0.0061	0.0030
8	0.0042	0.0040	0.0041	0.0039	0.0041	0.0038	0.0041	0.0037
10	0.0036	0.0038	0.0036	0.0037	0.0035	0.0036	0.0035	0.0035
20	0.0032	0.0032	0.0031	0.0031	0.0031	0.0031	0.0030	0.0029
50	0.0029	0.0029	0.0029	0.0029	0.0029	0.0028	0.0028	0.0030

From this simulation study we can conclude that the Bayesian method generally gives smaller false alarm rates than the classical method. This is the case for most of the choices of m and λ, except for $\lambda-$values of 2 (when m is small) and 10 (when m is large) when considering the false alarm rate. When $m=25$ the Bayesian method gives a false alarm rate slightly higher than the classical method for almost all λ values. When $m=30$ the Bayesian method gives a false alarm rate smaller than the classical method for all λ values. Where the false alarm rates obtained from the Bayesian method were generally closer to the expected nominal value for the false alarm rate, 0.0027. The results obtained by us for the classical methods are similar to those obtained by Chakraborti & Human (2008).

4 Example

Consider the following example from Montgomery (1996), Example 6-3 on page 277. Chakraborti & Human (2008) also considered this example. This example deals with the number of nonconformities observed in 26 successive samples of 100 printed circuit boards. The inspection unit is defined as 100 boards. The 26 samples contained 516 nonconformities, and λ is estimated by

$$\bar{\lambda} = \frac{516}{26} = 19.85.$$

It was found that units 6 and 20 were out-of-control, and they were eliminated after further investigation. Revised limits were calculated using the remaining samples, with $m = 24$ and $\sum_{i=1}^{m} x_i = 472$. The average number of nonconformities per inspection unit was recalculated as

$$\bar{\lambda} = \frac{472}{24} = 19.67.$$

Considering this example, we have $m = 24$ and $\sum_{i=1}^{m} x_i = 472$. As mentioned, we will use the Jeffreys prior for λ, $\pi_J(\lambda) \propto \lambda^{-\frac{1}{2}}$. This will result in a gamma posterior, $Gamma\left(\sum_{i=1}^{m} x_i + \frac{1}{2}, m\right)$. For this example the posterior distribution of λ will be a $Gamma(472.5, 24)$. Figure 2 shows the posterior distribution of λ. Figure 3 shows a bar graph of the predictive density function, $f\left(x_f \,|data\right)$, where x_f is the number of nonconformities in a future inspection unit. The predictive density will be used to obtain the control chart limits.

Summary statistics for the posterior distribution of λ:

mean = 19.69 standard deviation = 0.9057 median = 19.67

95% credibility interval = (17.95; 21.50).

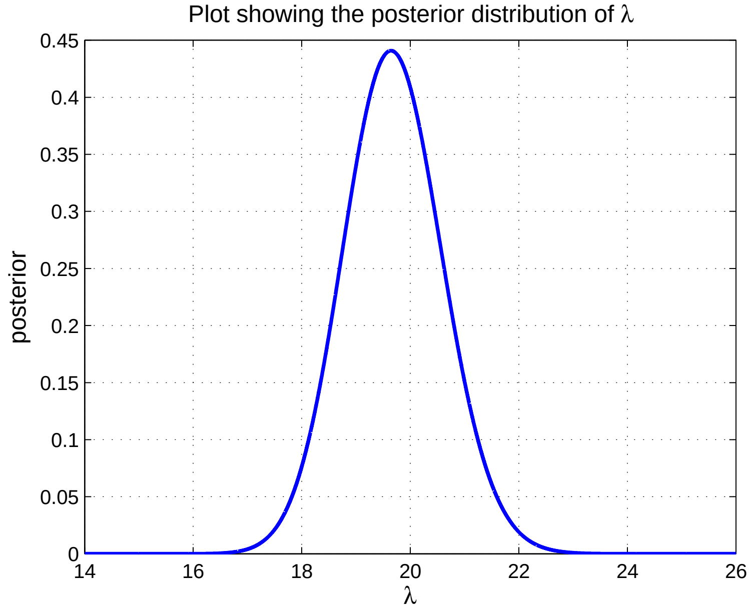

Figure 2: Posterior distribution of λ, when $m = 24$ and $\sum_{i=1}^{m} x_i = 472$.

For the observed value, $\sum_{i=1}^{m} = 472$, the chart's performance will be investigated by looking at the conditional average run length (CARL) and the conditional false alarm rate (CFAR). The control chart limits, conditional average run length and conditional average false alarm rate will be calculated using the classical (frequentist) method and using a Bayesian method. Chakraborti & Human (2008) determined these values using the classical method, the results are given in Table 4. We used a Bayesian method to determine these values, the results are also given in Table 4. From Chakraborti & Human (2008), we use $\lambda = 20$ to calculate CARL and CFAR.

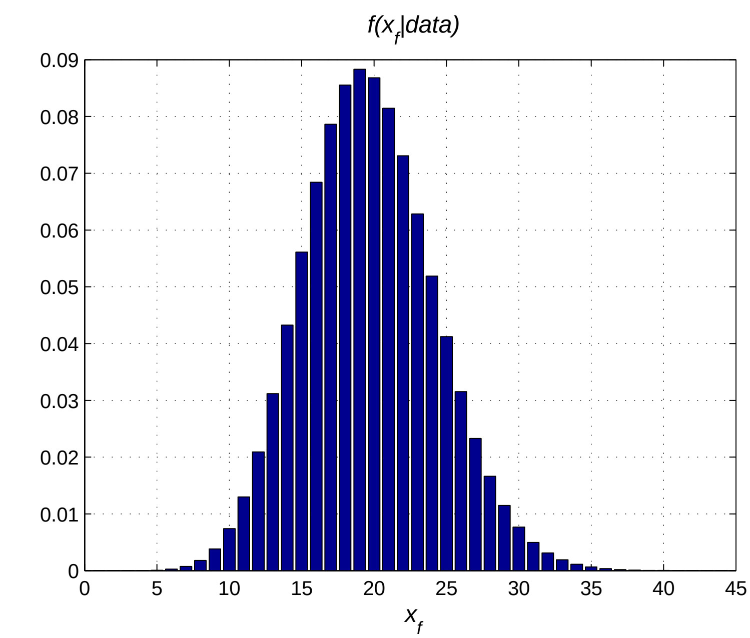

Figure 3: Bar graph of the predictive density of $f(x_f|data)$.

Table 4: Lower control limits, upper control limits, conditional average run lengths (CARL) and conditional false alarm rates (CFAR) for $m = 24$ and $\sum_{i=1}^{m} = 472$.

	LCL	UCL	CARL	CFAR
Frequentist	6	32	200.7	0.0050
Bayesian	8	35	267.50	0.0037

From Table 4, we see that the Bayesian method gives a wider interval than the classical method, this results in a larger value for CARL and a smaller value for CFAR. The conditional false alarm rate (CFAR) when the Bayesian method is used, is equal to 0.0037. This value is much closer to the nominal value of 0.0027, than the conditional false alarm rate (CFAR) of 0.0050 obtained from the classical method.

5 Tolerance Intervals

Tolerance intervals could be of interest in quality control. The construction of tolerance intervals to measure discrete quality characteristics has been one of the major tasks in developing quality control systems used in manufacturing and pharmaceutical sectors, (Wang & Tsung, 2009). A tolerance interval gives information about a certain proportion or more of the population, with a given confidence level. This proportion is also referred to as the content of a tolerance interval. Whereas a confidence interval gives information about an unknown parameter. The goal of a tolerance interval is to contain at least a specified proportion of the population with a specified degree of confidence.

The predictive density can be used to construct a tolerance. In this paper we will look at the $\pi-$expectation tolerance interval. Where π is the expected coverage of the interval. The π - expectation intervals focus on the prediction of one or a few future observations from the process. A tolerance interval can be one-sided or two-sided. The one-sided interval can take the form $(-\infty, U)$ or (L, ∞), where U is called a one-sided upper tolerance limit and L a one-sided lower tolerance limit. The two-sided interval can take on two different types. From Krishnamoorthy & Mathew (2009) the one is constructed such that it would contain at least a proportion π of the population with confidence level $1-\alpha$, and the other type is constructed such that it would contain at least a proportion π of the centre of the population with confidence level $1-\alpha$. The latter is referred to as an equal-tailed tolerance interval. The Jeffreys prior will be used for the Bayesian method.

Consider the following study for the interval estimation of the 95^{th} percentile of the Poisson distribution. For given x, $x = 0, 1, \ldots 50$, 10 000 values are simulated from the posterior distribution of λ. For each value of λ, the corresponding 95^{th} percentile (P_{95}) of the Poisson distribution is found. From the sorted 10 000 values of P_{95} the lower limit (2.5%) and upper limit (97.5%) is found in the case of the two-sided interval. For given λ, $\lambda = 1, \ldots, 15$, the probabilities for all values of x which yielded an interval which contains the true λ are added to obtain the coverage probability. The results are given in Figure 4.

From 4 it can be seen that the coverage rates are most of the time at or above the nominal value of 0.95. When $\lambda = 3$, 12.5 and 13 the coverage rates are below 0.95 for the two-sided case, and when $\lambda = 4$, 4.5, 8.5, 9 and 10.5 the coverage rates are below 0.95 for the one-sided case.

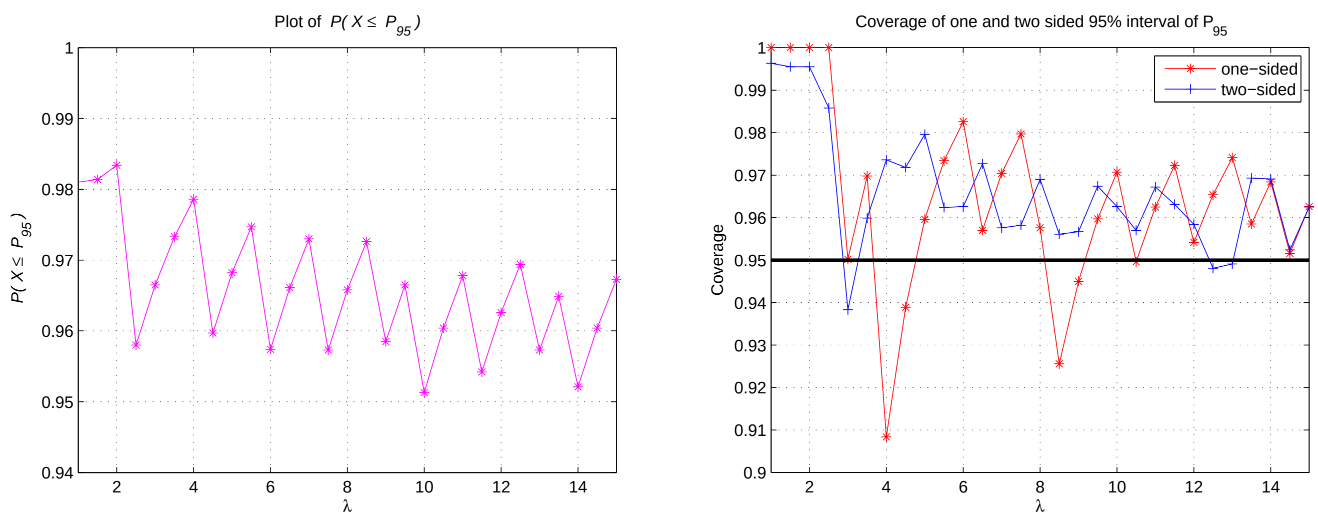

Figure 4: Coverage of the one- and two-sided 95% interval for P_{95}.

6 Conclusion

The usual operation of the c - chart was extended by introducing a Bayesian approach for the c - chart. From an extended simulation study, using different values of λ and number of inspection units, we conclude that the suggested Bayesian approach gives larger values for the average run length and smaller values for the false alarm rate. A smaller value for the false alarm rate is desired, since the false alarm rate is the probability of a signal being given by the control chart when the process is actually in-control. The false alarm rates obtained from the Bayesian method were generally closer to the expected nominal value for the false alarm rate, 0.0027. The Bayesian method generally had wider control limits.

Bayarri & García-Donato (2005) give the following reasons for recommending a Bayesian analysis:

- Control charts are based on future observations, and Bayesian methods are very natural for prediction;
- uncertainty in the estimation of the unknown parameters is adequately handled;
- implementation with complicated models and in a sequential scenario poses no methodological difficulty, the numerical difficulties are easily handled via Monte Carlo methods;
- objective Bayesian analysis is possible without introduction of external information other than the model, but any kind of prior information can be incorporated into the analysis, if desired.

A Bayesian tolerance interval for the Poisson distribution was introduced, and from the simulation studies it was seen that the coverage rates obtained for one-sided and two-sided intervals were relatively good. For the two-sided and one-sided intervals the coverage rates were most of the time at or above 0.95, except in a few cases.

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